ODD GRACEFUL LABELINGS OF CYCLIC SNAKES

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ABSTRACT

A difference vertex labeling of a graph G an assignment ϕ of labels to the vertex of G that induces for each edge xy the weight $|\phi(x) - \phi(y)|$. A difference vertex labeling ϕ of a graph G of size n is odd-graceful if ϕ is an injection from $V(G)$ to $\{0, 1, 2, \ldots, 2q-1\}$ such that the induced weights are $\{1, 3, 5, \ldots, 2q-1\}$. In this paper, we present odd graceful labelings of some graphs. In particular we show, odd graceful labelings of the kC_4- *snakes* (for the general case), kC_8 *and* kC_{12}- *snakes* (for even case). We also prove that the linear kC_n- *snakes* is odd graceful if and only if n and k are even.

Key words : Odd Graceful labelings, cycles, snakes.

1. Introduction

A graph G of size q is odd-graceful, if there is an injection ϕ from $V(G)$ to $\{0, 1, 2, \ldots, 2q-1\}$ such that, when each edge xy is assigned the label or weight $|\phi(x) - \phi(y)|$, the resulting edge labels are $\{1, 3, 5, \ldots, 2q-1\}$. This definition was introduced in 1991 by Gnanajothi [2] who proved that the class of odd graceful graphs lies between the class of graphs with α-labelings and the class of bipartite graphs. Gnanajothi [2] proved that every cycle graph C_n is odd graceful if and only if n is even. She proved that the graphs obtained by joining a single pendant edge to each vertex of C_n are odd graceful if and only if n is even. Moussa and Badr [10] generalized Gnanajothi's result on cycle C_n by showing that the graphs obtained by joining m pendant edges to each vertex of C_n are odd graceful if and only if n is even.

Rosa [9] has defined a triangular snake (or Δ-snake) as a connected graph in which all block-cutpoint graph is a path, and conjectured that Δ_n-snake (a Δ-snake with blocks) is graceful for $n \equiv 0$ or 1(mod 4) and is nearly graceful otherwise. In 1989 Moulton [13] has proved Rosa's conjecture but using instead of nearly graceful labelings an stronger labeling named almost graceful.

Barrientos [14] proved that the kC_4-*snacke* (for the general case) , kC_8-*snakes* and kC_{12}-*snakes* (for the even case) are graceful. He also established some conditions to obtain graceful labelings of kC_{4n}-*snakes*. Moreover, of the linear kC_6-*snacke*, we show a graceful labeling when k is even and nearly graceful labeling when k is odd. In our study we show, odd graceful labelings of the kC_4- *snakes* (for the general case), kC_8 *and* kC_{12}- *snakes* (for even case). We also prove that the linear kC_n- *snakes* is odd graceful if and only if n is even. In section 2, we introduce a special method which proves that the cyclic graph is odd graceful. In section 3, we introduce odd graceful labelings of the kC_4- *snakes* (for the general case), kC_8 *and* kC_{12}- *snakes* (for even case). We also prove that the linear kC_n- *snakes* is odd graceful if and only if n and k are even.

2. Odd Graceful labelings of the Cyclic Graph C_n

We are introducing a special method for labeling the vertices of the cyclic graph C_n. Firstly, we show how to use this method in order to prove that the cyclic graph C_n is odd graceful. Secondly, we use this method for proving other further theorems. We denote the total number of edges of the cyclic graph C_n by q.

Theorem 2.1. The cyclic graph C_n is odd graceful if and only if n is even ($n \geq 4$).

Proof.

Draw the cyclic graph as the graph which consists of two paths as a left path (L) $u\ u_1\ u_2\ u_3\ \ldots u_{(n/2)-1}$ and a right path (R) $v_1\ v_2\ v_3\ \ldots\ v = v_{n/2}$. In order to get the cyclic graph, connect the vertex u with the vertex v_1 and connect the vertex v with the vertex $u_{(n/2)-1}$ (Fig.1(a)). Clearly C_n has n vertices and q edges such that $q = n$.

Let us consider the following numbering ϕ of the vertices of C_n.

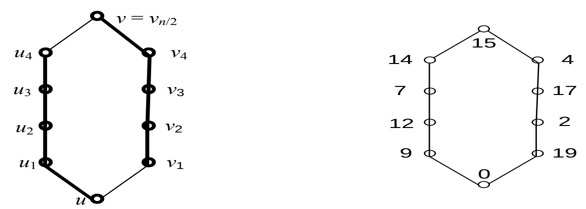
$\phi(u) = 0$

$\phi(u_i) = q - i$, (i odd) $i = 1, 2, \ldots, (n/2)-1$

$\phi(u_i) = q + i$, (i even) $i = 1, 2, \ldots, (n/2)-1$

$\phi(v_i) = 2q - i$, (i odd) $i = 1, 2, \ldots, (n/2)-1, n/2$

$\phi(v_i) = i$, (i even) $i = 1, 2, \ldots, (n/2)-1, n/2$



a b

Figure 1

(a)
$$\max_{v \in V} \phi(v) = \max\left\{0, \max_{\substack{i\ odd \\ 1 \le i \le \frac{n}{2}-1}} q - i, \max_{\substack{i\ even \\ 1 \le i \le \frac{n}{2}-1}} q + i, \max_{\substack{i\ odd \\ 1 \le i \le \frac{n}{2}}} 2q - i, \max_{\substack{i\ even \\ 1 \le i \le \frac{n}{2}}} i \right\}$$

$= 2q - 1$, the maximum value of all odds. Thus $\phi(v) \in \{0, 1, 2, \ldots, 2q - 1\}$.

(b) Clearly ϕ is one – to – one the vertex set of C_n.

(c) It remains to show that the labels of the edges of C_n are all the integers of the interval $[1, 2q\text{-}1]$.

The rang of $|\phi(v_i) - \phi(v_{i+1})| = \{2q - 2i - 1 : i = 1, 2, \ldots, \frac{n}{2} - 1\}$

$= \{ 2q \text{-}3, 2q\text{-}5, \ldots, q + 1\}$

The rang of $|\phi(u_i) - \phi(u_{i+1})| = \{2i + 1 : i = 1, 2, 3, \ldots, \frac{n}{2} - 2\}$

$= \{ 3, 5, \ldots, q\text{-}3\}$

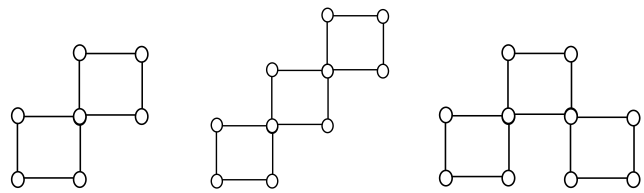
The edges uu_1, uv_1 and $v\,u_{(n/2)-1}$ are labeling by q-1, $2q$-1 and 1 respectively.

Hence $\{|\phi(u) - \phi(v)| : uv \in E\} = \{1, 3, 5, \ldots, 2q - 1\}$ so that C_n is odd graceful for n is even. ■

The odd graceful numbering of C_{10} is displayed in Fig. 1(b)

3. Odd Graceful Labelings Of Cyclic Snakes

Consider the following generalization of triangular snakes. For a kC_n- *snake*, we understood a connected graph in which the k blocks are isomorphic to the cycle C_n and the block-cutpoint graph is a path. We are interested in study under what conditions are they odd-graceful. In Figure 2 we show the cases that appears now.



(a) (b) (c)

Figure 2

On Figure 2(a) we show the unique $2C_4$- *snakes*. Beginning with this graph and using the fact that we only have two vertices where to connect the new copy of C_4, one to distance 1 from the cut vertex and the other at distance 2, we must obtain two non isomorphic $3C_4$- *snakes*, showed in 2(b) and 2(c). In general, following both schemes we may obtain at most 2^{k-2} different kC_4- *snakes*, some of them are isomorphic.

To construct the kC_4- *snakes* we may apply these schemes and since our graph is bipartite graph (one partite set has black vertices and the other has white vertices) it is possible to embed it, on an square grid as is showed in Figure 3.

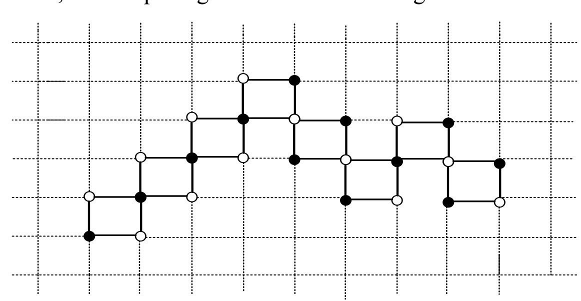


Figure 3

Consider the following numbering of the vertices of C_4 showed in Figure 4. Assuming that x and y are non negative integers and that $x < y$ then, the weights induced are $y - x - 2$, $y - x$, $y - x + 4$, and $y - x + 2$.

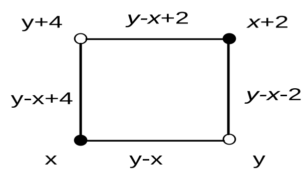


Figure 4

Put black vertices in an string, ordered by diagonals from left to right and inside each diagonal from bottom to top, assign to them from an arithmetic progression of difference 2 which first term is zero, counting until the last black vertex has been numbered. Similarly, put white vertices on an string, ordered for diagonals from right to left and inside each diagonal from bottom to top. Starting with first diagonal assign numbers from an arithmetic progression of difference 4 which first term is one more than the last integer used in the previous assignment, but every first vertex labeling in each diagonal (not the first vertex of the first diagonal) increase by 2 , continuing until the last white vertex has been numbered.

The labeling over each copy of C_4 is of the same kind that the used on Figure 4, and is possible to see that the complete labeling is an odd graceful labeling of kC_4-*snake*. Hence, we have proved the following theorem.

Theorem 2. The kC_4-*snake* has an odd graceful labeling.

In Figure 5 we show the odd graceful labeling (obtained from the theorem1) of the $8C_4$-*snake*.

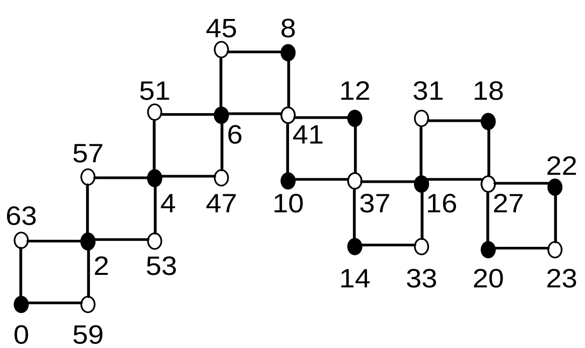


Figure 5

Let $G = 2C_n$-snake (n is even), G has two blocks and contains only one cut-vertex.

Take one of these blocks, there exist $\left\lceil \frac{n}{2} \right\rceil$ different ways to stick a new copy of C_n to G, that depends of the distance that separate the cut-vertex of the block selected and the vertex taken to stick the new block.

Consider the path P of minimum length that contains all the cut-vertices of G. Beginning in one of its extremes, is possible construct an string of $k - 2$ integers such that, any of this integers is the distance between two consecutive cut-vertices of G on the path P. Since P is of minimum length, the integer on the string are taken from $E = \{1, 2, \ldots, \frac{n}{2}\}$. Hence, any graph $G = kC_n$-*snake*, is represented by an string. Until now, this representation is not unique, because it depends of the extreme of P taken, but considering the strings obtained for both extremes as the same, we avoid the problem.

For instance, the string (from left to right) of the $8C_4$-*snake* on Figure 4 is 2, 2, 1, 2, 1, 1. In the case of kC_3-*snake*, the sequences is 1, 1, …, 1. If the string of a given kC_n-*snake* is $\frac{n}{2} \ldots \frac{n}{2}$, we say that kC_n-*snake* is linear.

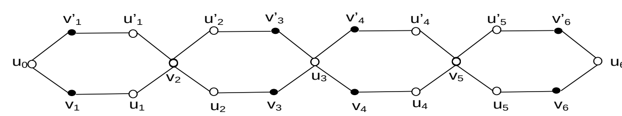


Figure 6

Theorem 3. The linear kC_n-*snake* has an odd graceful labeling if and only if n and k are even.

Proof.

Let kC_n-*snake* is the graph which has q edges and n vertices. The graph kC_n-*snake* consists of the vertices $(u_0, u_1, u_2, u_3, \ldots, u_l)$, $(v_1, v_2, v_3, \ldots, v_m)$, $(u'_1, u'_2, u'_3, \ldots, u'_{l-1})$ and $(v'_1, v'_2, v'_3, \ldots, v'_m)$ as are shown in Figure 6 ($4C_4$-*snake*) .

Let us consider the following numbering ϕ of the vertices of kC_n-*snake* (n and k are even)

$\phi(u_i) = 2i, \qquad i = 0, 1, 2, \ldots, l$

$\phi(v_i) = 2q - 2i + 1, \qquad i = 1, 2, \ldots, m$

$\phi(u'_i) = q + 2i, \qquad i = 1, 2, \ldots, l-1$

$\phi(v'_i) = q - 2i + 1, \qquad i = 1, 2, \ldots, m$

(a) $$\underset{v \in V}{\text{Max}} \; \phi(v) = \max\left\{ \max_{0 \le i \le l} 2i \,,\, \max_{1 \le i \le m} 2q - 2i + 1, \; \max_{1 \le i \le l-1} q + 2i \,,\, \max_{1 \le i \le m} q - 2i + 1 \right\}$$

= $2q - 1$, the maximum value of all odds. Thus $\phi\ (v) \in \{0, 1, 2,\ldots,2q - 1\}$.

(b) Clearly ϕ is one – to – one the vertex set of kC_n-*snake*.

(c) It remains to show that the labels of the edges of kC_n-*snake* are all the integers of the interval $[1, 2q-1]$.

The rang of $|\phi(u_i) - \phi(v_i)| = \{2q - 4i + 1 : i = 1,2,...,m\}$

$= \{ 2q - 3, 2q - 7, \ldots, 2q - 4m + 1\}$

The rang of $|\phi(u_i) - \phi(v_{i+1})| = \{2q - 4i - 1 \;\; : i = 0,1,2,3,...,m\}$

$= \{2q-1, 2q-5, \ldots, 2q - 4m - 1\}$

The rang of $|\phi(u_i) - \phi(v_i)| = \{4i - 1 \;\; : i = 1,2,3,...,m - 1\}$

$= \{3, 7, \ldots, 2q - 4m - 5\}$

The rang of $|\phi(u'_i) - \phi(v'_{i+1})| = \{4i + 1 \;\; : i = 1,2,3,...,m - 1\}$

$= \{5, 9, \ldots, 4m - 3\}$

The rang of $|\phi(u'_i) - \phi(v_{i+1})| = \{q - 4i - 1 \;\; : i = 1,2,3,...,m - 1\}$

$= \{q-5, q-9, \ldots, q - 4m + 3\}$

The rang of $|\phi(u_i) - \phi(v'_i)| = \{q - 4i + 1 \;\; : i = 1,2,3,...,m - 1\}$

$= \{q-3, q-7, \ldots, q - 4m + 5\}$

The edges $u_0v'_1$ *is* labeling by q-1.

Hence $\{|\phi(u) - \phi(v)| : uv \in E\} = \{1,3,5,...,2q - 1\}$ so that kC_n-*snake* is odd graceful for (n and k are even) . ■

In Figure 7 we show the odd graceful labeling (obtained from the theorem3) of the $4C_8$-*snake*.

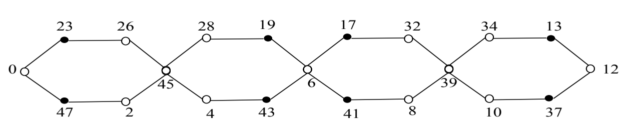


Figure 7

If G is a cyclic snake whose string contains only even numbers, we say that G is an even cyclic snake. In the follow, we only work with even cyclic snakes. The next theorem focus in even kC_8 and kC_{12}-*snakes*. Consider the labelings of C_8 and C_{12} showed in Figure 8. Observe that if t = 8 or t = 12, the numberings are odd graceful labelings of C_8 and C_{12}, respectively. In the next theorem we introduce odd graceful labelings of kC_8 and kC_{12}-*snakes.*

Theorem 4. The even kC_8 and kC_{12}- snakes are odd-graceful graphs.

Proof:

Let G be any even kC_{12}- snake, where n = 8, 12. Then its string is of the form $d_1, d_2, \ldots, d_{k-2}$, $d_i \in \{2, 4\}$ if $n = 8$ or $d_i \in \{2, 4, 6\}$ if $n = 12$. Denote by $B_{1,n}, B_{2,n}, \ldots, B_{k,n}$ the consecutive blocks of G. First, we label the blocks of G :

block	labeling used	translation
$B_{1,n}$	Type $n/2$, $t = 2nk - 1$	
$B_{i+1,n}$	Type d_i, $t = 2n(k - i)$ -1	$(2n - 4)i / 2$
$B_{k,n}$	Type $n/2$, $t = 2n$ - 1	$(2n - 4)(k - 1) / 2$

Second, the weights on the block $B_{i,n}$ are $n(k - i) + 1$, $n(k - i) + 2$, … , $n(k - i) + n$. Science $1 < i < k$, the weights on G are 1, 2, …, nk. Third, the labels used are in the set $\{0, 1, 2, \ldots, nk\}$. Finally, $B_{i,n}$ and $B_{i+1,n}$ are connected by its vertices of label $(n - 2)\, i/2$, obtaining the graph G with an odd graceful labeling. ■

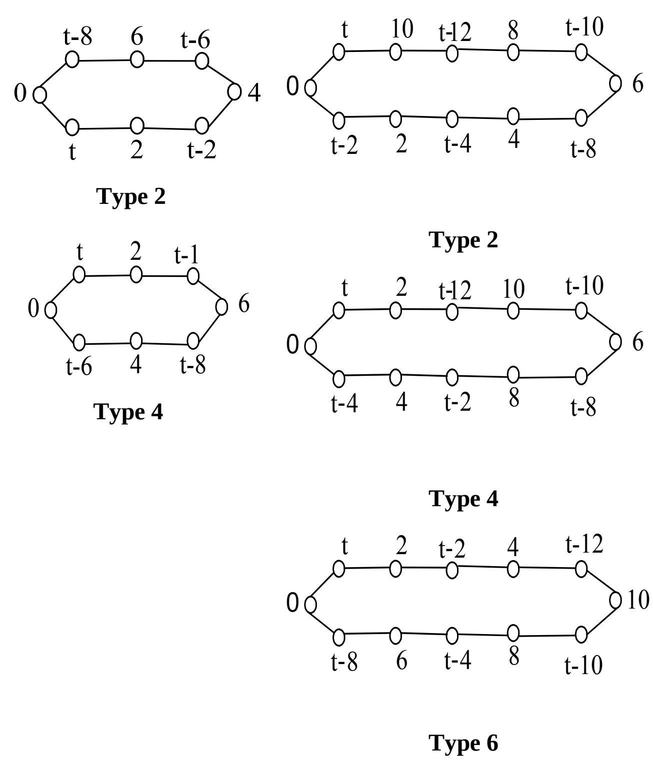


Figure 8 : labeling of kC_8 and kC_{12}

4. Conclusion

Since labeled graphs serve as practically useful models for wide ranging applications such as communications network, circuit design, coding theory, radar, astronomy, x-ray and crystallography. It is desired to have generalized results or results for a whole class, if possible. This work has presented odd graceful labelings of some graphs. In particular we show, odd graceful labelings of the kC_4- *snakes* (for the general case), kC_8 *and* kC_{12}- *snakes* (for even case). We also proved that the linear kC_n- *snakes* is odd graceful if and only if n and k are even.

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